

Accumulation on the boundary for one-dimensional stochastic particle system

V. A. Malyshev, A. A. Zamyatin

Abstract

We consider infinite particle system on the positive half-line moving independently of each other. When a particle hits the boundary it immediately disappears, and the boundary moves to the right on some fixed quantity (particle size). We study the speed of the boundary movement (growth). Possible applications - dynamics of the traffic jam growth, growth of thrombus, epitaxy. Nontrivial mathematics is related to the correlation between particle dynamics and boundary growth.

1 Introduction

Hitting the boundary by the random walk is a classical problem of probability theory. More complicated is the problem of hitting the moving boundary, see [1]. Here we consider the case when the movement of the boundary is correlated with the movement of the particles. Namely, we consider infinite number of particles, performing random walks on the positive half-line, which stick to the boundary when hit it. Moreover, when a particle hits the boundary, the boundary moves to the right on some quantity, “size” of the particle. That is the boundary moves due to accumulation of particles on it. As far as we know, such problems were not considered earlier.

Similar problems one can often encounter in applications - growth of thrombus, epitaxy and other methods of coating the surface with metal particles. In many cases the effect of correlation between the movements of particles and the boundary can be neglected. Here, on the contrary, we consider the influence of this correlation on the growth speed of the boundary. Such effects can be observed while quick formation of traffic jams.

In the paper we present two results. First one for nonzero drift of the particles, second - for zero drift. In the first case one can find exact asymptotic rate of growth, in the second case one can get only the order of growth.

2 Formulation of the problem and the results

We consider infinite system of particles on the half-line, which, while hitting the boundary, adhere to it (accumulate on it). As a result the boundary moves to the right. Now we give exact formulation.

Assume that at time 0 on the half-line R_+ the random configuration of particles at the points

$$0 \leq x_1 = x_1(0) < \dots < x_n = x_n(0) < \dots$$

is distributed as the point Poisson process with density λ .

Each particle, before hitting the boundary, moves independently of other particles as

$$x_i(t) = x_j - vt + w_j(t) \tag{1}$$

where $v \geq 0$ is the constant drift, and $w_i(t)$ is the standard Wiener process with zero mean. At the hitting moment the particles are absorbed by the boundary and disappear, moreover the boundary moves to the right on some δ , the “particle size”.

More exactly, the boundary which at time $t = 0$ coincides with point $0 \in R_+$, also moves. The law $\xi(t)$ of its movement models the sticking of particles to the boundary. Namely, $\xi(0) = 0$, and $\xi(t)$ is a non-decreasing point-wise constant function with jumps

$$\xi(t_j + 0) = \xi(t_j) + \delta k_j$$

at random time moments

$$0 \leq t_1 < \dots < t_j < \dots$$

where $\delta > 0$ is some constant, the “particle size”, the moments t_j and the non-negative integers k_j being defined by the following recurrent procedure. Let t_1 be the first instant, when one of the particles hits 0, then k_1 is the minimal integer $k > 0$ such that in the interval $(0, k\delta]$ at this moment t_1 there are not more than $k - 1$ particles. For example, $k = 1$ if and only if at time t_1 there are no particles inside the interval $(0, \delta]$. Further on by induction, t_{n+1} is defined as the first moment when one of the remaining particles hits the point $(k_1 + \dots + k_n)\delta$, and k_{n+1} is defined correspondingly as the minimal integer $k > 0$ such that inside the interval $(\xi(t_n), k\delta]$ at time t_{n+1} there is no more than $k - 1$ particles.

Below the following statement will be proved.

Lemma 1 *If $\lambda < \delta^{-1}$, then with probability 1 all k_i and t_i are finite, and moreover $t_i \rightarrow \infty$. On the contrary, if $\lambda > \delta^{-1}$, the boundary reaches infinity for finite time with positive probability.*

Now we can formulate the main result. Let $N = N(t)$ be the number of particles, absorbed by the boundary to time t , then $\xi = \xi(t) = \delta N(t)$ is the coordinate of the particle at time t .

Theorem 2 *Let $\lambda < \delta^{-1}$. If $v > 0$, the particle movement is asymptotically linear, that is as $t \rightarrow \infty$ a. s.*

$$\frac{\xi(t)}{t} \rightarrow V = v \frac{\delta \lambda}{1 - \delta \lambda}$$

If $v = 0$, then for t sufficiently large

$$C_0 \leq \frac{MN(t)}{\sqrt{t}} \leq C$$

for some constants

$$\lambda \sqrt{\frac{2}{\pi}} < C_0 < C < \infty$$

Remark 3 *We will show later that under the condition that the boundary does not move at all and $v = 0$, then the mean number of particles hitting the boundary during time t equals $\lambda \sqrt{\frac{2t}{\pi}}$. From the second assertion of the theorem it follows in particular that the mean number of particles hitting the boundary is asymptotically greater. We conjecture that $\frac{MN(t)}{\sqrt{t}}$ tends to some constant as $t \rightarrow \infty$, and moreover the distribution of the random variable $\frac{N(t)}{\sqrt{t}}$ tends to some continuous distribution as $t \rightarrow \infty$.*

3 Proofs

Proof of Lemma 1 For a given configuration $X = \{x_i(0)\}$ denote $p(X, n)$ the probability that the first particle hitting the boundary is the particle $x_n(0)$.

To prove the first part of the Lemma introduce the following auxiliary model. At time $t = 0$ for a given configuration on the positive axis we add Poisson configuration on the negative axis Y , having the same density λ . Then we will get Poisson configuration on all line R , all these particles have red color. Moreover, there is one more (blue) particle, which we put with probability $p(X, n)$ at the point $x_n(0) > 0$. The particles move according to (1), independently of each other. At time $t = 0$ the boundary is at the point $\xi'(0) = 0$. The random variable t'_1 is defined as the first moment when the blue particle hits 0, and k'_1 are defined as earlier. The difference from the basic model is that the boundary does not influence other particles at all. Thus at any moment t the red particles $y_j(t)$, $-\infty < j < \infty$, have Poisson field distribution (Doob-Dobrushin theorem, [3]). Moreover at random time t'_1 the red particles have Poisson field distribution.

Let $\xi'(t)$ be the position of the boundary at time t . Then for all t a. s. $\xi'(t) \geq \xi(t)$. In particular, $k'_1 \geq k_1$ a. s. In fact, let Ω be the probability space of the auxiliary model. The basic model is defined on the same probability space, as it can be obtained from the auxiliary model by deleting the glue particle and the particles situated to the left of 0. Hence in the auxiliary model the number of collisions with the boundary is not less than in the basic model.

Let us prove that k'_1 is finite a. s. for any n . The random variable k'_1 is defined by the stopping time of the following Markov chain m_1, \dots, m_i, \dots with discrete time and state space Z_+ . The state 0 is an absorbing state. Denote m_1 the number of particles in the interval $[0, \delta]$ at time t_1 , and m_2 the number of particles in the interval $[\delta, (m_1 + 1)\delta]$, if $m_1 > 0$, m_3 the number of particles in the interval $[(m_1 + 1)\delta, (m_1 + 1)\delta + m_2\delta]$, if $m_1 > 0$, $m_2 > 0$, and so on. The sequence m_i constitute time homogeneous Markov chain with transition probabilities

$$p_{m_i m_{i+1}} = \frac{(m_i \lambda \delta)^{m_{i+1}}}{m_{i+1}!} e^{-m_i \lambda \delta}$$

The boundary stops when $m_i = 0$. One step increment is

$$M(m_{i+1} - m_i | m_i = k) = k(\lambda \delta - 1)$$

The chain hits 0 with probability 1, if $\lambda \delta - 1 < 0$.

By induction, using similar trick with appending a new blue particle at any moments t'_i , the finiteness of all k'_i follows.

The second statement of the Lemma is not used in the proof of the Theorem, and it will be convenient to prove it at the end of the paper after the proof of the Theorem.

Idea of the proof of the Theorem One can write down two equalities relating random variables $\xi(t)$ and $N(t)$:

$$\xi(t) = N(t)\delta \quad (2)$$

$$N(t) = n_0(t, \xi(t)) + n_1(t, \xi(t)) \quad (3)$$

where $n_0(t, \xi)$ is the number of particles which were at time 0 inside the interval $[0, \xi]$ and were absorbed by the boundary before time t , $n_1(t, \xi)$ is the number of particles which were at time 0 to the right of the point ξ and were absorbed by the boundary before time t . We can write the following equation

$$\delta^{-1}\xi(t) = n_0(t, \xi(t)) + n_1(t, \xi(t))$$

It will appear useful if we shall have good expressions for n_0 and n_1 .

The first result of the theorem can be explained for the case when for nonzero drift there is no fluctuations. More exactly, the particles, situated initially at the points $\lambda^{-1}k$ with integer k , move with constant velocity v . Then from equations (2),(3) we have

$$\xi = N\delta, N = [\xi\lambda] + [\lambda vt] \quad (4)$$

Hence

$$N - [\lambda N\delta] = [\lambda vt]$$

$$N = N(t) = \frac{[\delta \lambda vt]}{1 - \frac{1}{N}[\lambda N\delta]} \sim \frac{\delta \lambda vt}{1 - \delta \lambda}$$

for large t . Then

$$\frac{\xi(t)}{t} \rightarrow V = v \frac{\delta \lambda}{1 - \delta \lambda}$$

3.1 The case of nonzero drift

Fix arbitrary $\epsilon > 0$, so that $(\lambda + \epsilon)\delta < 1$. Let $c_\epsilon = (1 - (\lambda + \epsilon)\delta)^{-1}(\lambda + \epsilon)(v + \epsilon)$. The event $A(t) = \{N(t, \omega) > c_\epsilon t\}$ can be written as the union

$$A(t) = \cup_n B(n, t)$$

where $B(n, t)$ is the event that to the time t the boundary has absorbed exactly n particles, situated initially at the interval $(0, c_\epsilon t\delta + (v + \epsilon)t)$ and more than $m = c_\epsilon t - n$ particles, situated initially at the interval $(c_\epsilon t\delta + (v + \epsilon)t, \infty)$. Denote $B_\epsilon(t)$ the union of events $B(n, t)$ with $n \geq (\lambda + \epsilon)(\delta c_\epsilon t + (v + \epsilon)t) = c_\epsilon t$. The event $B_\epsilon(t)$ has exponentially small probability, as it belongs to the event $C_\epsilon(t)$ that at time 0 in the interval

$(0, c_\epsilon t\delta + (v + \epsilon)t)$ there were not less than $(\lambda + \epsilon)(\delta c_\epsilon t + (v + \epsilon)t) = c_\epsilon t$ particles. The probability of the latter event can be estimated via Poisson distribution:

$$P(C_\epsilon(t)) \leq C_0 e^{-h(\epsilon)t} \quad (5)$$

where constant C_0 depends on λ and ϵ , $h(\epsilon) > 0$.

Denote $D_\epsilon(t)$ the union of the events $B(n, t)$ with $n < (\lambda + \epsilon)(\delta c_\epsilon t + (v + \epsilon)t) = c_\epsilon t$. The event $D_\epsilon(t)$ belongs to the event that at least one of the particles, situated initially in $(c_\epsilon t\delta + (v + \epsilon)t, \infty)$, before t will be absorbed by the boundary. The latter event in its turn belongs to the event $J_\epsilon(t)$, that at least one of the particles, situated initially in $(c_\epsilon t\delta + (v + \epsilon)t, \infty)$, had been to the left of the point $c_\epsilon t\delta$ at least once before t . Thus,

$$A(t) \subset C_\epsilon(t) \cup J_\epsilon(t)$$

The probability of the event $J_\epsilon(t)$ can be estimated from above by the expectation of the number of particles, situated initially in $(c_\epsilon t\delta + (v + \epsilon)t, \infty)$, and visiting at least once the left side of the point $c_\epsilon t\delta$ before t

$$P(J_\epsilon(t)) \leq \lambda \int_{c_\epsilon t\delta + (v + \epsilon)t}^{\infty} p(t, x) dx$$

where $p(t, x)$ is the probability that the particle started at x , $x > c_\epsilon t\delta + (v + \epsilon)t$, during t will visit the left side of the point $c_\epsilon t\delta$. More exactly,

$$p(t, x) = P(\min_{s \leq t} (x - vs + w(s)) \leq c_\epsilon t\delta) \quad (6)$$

From the definition of Wiener process it follows that

$$\begin{aligned} P(\min_{s \leq t} (x - vs + w(s)) \leq c_\epsilon t\delta) &= P(\min_{s \leq t} (-vs + w(s)) \leq -x + c_\epsilon t\delta) \\ &= P(\max_{s \leq t} (vs + w(s)) \geq x - c_\epsilon t\delta) \end{aligned}$$

To find $p(t, x)$ we will use the well known formula for the distribution of the maximum of Wiener process with drift (see for example, [1], p. 935)

$$P(\max_{s \leq t} (vs + w(s)) \geq x) = 1 - \Phi\left(\frac{x - vt}{\sqrt{t}}\right) + e^{2vx} \Phi\left(\frac{-x - vt}{\sqrt{t}}\right) \quad (7)$$

for $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-y^2/2} dy$. Then

$$\begin{aligned} P(J_\epsilon(t)) &\leq \lambda \int_{c_\epsilon t\delta + (v + \epsilon)t}^{\infty} \bar{\Phi}\left(\frac{x - c_\epsilon t\delta - vt}{\sqrt{t}}\right) dx \\ &\quad + \lambda \int_{c_\epsilon t\delta + (v + \epsilon)t}^{\infty} e^{2vx} \Phi\left(\frac{-x + c_\epsilon t\delta - vt}{\sqrt{t}}\right) dx \\ &= I_1(\epsilon, t) + I_2(\epsilon, t) \end{aligned} \quad (8)$$

where $\bar{\Phi} = 1 - \Phi$. Change of variable $z = \frac{x - c_\epsilon t\delta - vt}{\sqrt{t}}$ in the first integral in the formula (8) gives

$$I_1(\epsilon, t) = \lambda \sqrt{t} \int_{\epsilon\sqrt{t}}^{\infty} \bar{\Phi}(z) dz \leq \lambda \sqrt{\frac{t}{2\pi}} e^{-\frac{\epsilon^2 t}{2}}$$

with $\epsilon\sqrt{t} > 1$.

Change of variables $z = \frac{x - c_\epsilon t\delta + vt}{\sqrt{t}}$ in the second integral in (8) gives

$$\begin{aligned} I_2(\epsilon, t) &= \lambda \sqrt{t} e^{-2v^2 t} \int_{(2v + \epsilon)\sqrt{t}}^{\infty} e^{2v\sqrt{t}z} \Phi(-z) dz \\ &= \lambda \sqrt{t} e^{-2v^2 t} \int_{(2v + \epsilon)\sqrt{t}}^{\infty} e^{2v\sqrt{t}z} \bar{\Phi}(z) dz \end{aligned}$$

For $\epsilon\sqrt{t} > 1$ we have

$$\begin{aligned} I_2(\epsilon, t) &\leq \lambda \sqrt{\frac{t}{2\pi}} \int_{(2v+\epsilon)\sqrt{t}}^{\infty} e^{-(z^2/2 - 2vz\sqrt{t} + 2v^2t)} dz \\ &= \lambda \sqrt{\frac{t}{2\pi}} \int_{(2v+\epsilon)\sqrt{t}}^{\infty} e^{-(z-2v\sqrt{t})^2/2} dz \\ &= \lambda \sqrt{t} \bar{\Phi}(\epsilon\sqrt{t}) \leq \lambda \sqrt{\frac{t}{2\pi}} e^{-\frac{\epsilon^2 t}{2}} \end{aligned}$$

Thus one can see that for $\epsilon\sqrt{t} > 1$

$$P(J_\epsilon(t)) \leq \lambda \sqrt{\frac{2t}{\pi}} e^{-\frac{\epsilon^2 t}{2}} \quad (9)$$

As the event $A(t)$ belongs to the union of the events $C_\epsilon(t)$ and $J_\epsilon(t)$, then using the estimates (5) and (9), we get that for any sufficiently small $\epsilon > 0$

$$P\left(N(t) > \frac{(\lambda + \epsilon)(v + \epsilon)t}{1 - (\lambda + \epsilon)\delta}\right) = O\left(e^{-\alpha_1(\epsilon)t}\right) \quad (10)$$

for some $\alpha_1(\epsilon) > 0$.

Consider now the event $A'(t) = \{N(t, \omega) < c'_\epsilon t\}$, where $c'_\epsilon = (1 - (\lambda - \epsilon)\delta)^{-1}(\lambda - \epsilon)(v - \epsilon)$. Define the event $C'_\epsilon(t)$, that at time 0 inside the interval $(0, c'_\epsilon t\delta + (v - \epsilon)t)$ there were not more than $(\lambda - \epsilon)(\delta c'_\epsilon t + (v - \epsilon)t) = c'_\epsilon t$ particles and let $\bar{C}'_\epsilon(t)$ be the complement to $C'_\epsilon(t)$. We can write the event $A'(t)$ as the union

$$A'(t) = B'_\epsilon(t) \cup D'_\epsilon(t)$$

where $B'_\epsilon(t) = A'(t) \cap C'_\epsilon(t)$ and $D'_\epsilon(t) = A'(t) \cap \bar{C}'_\epsilon(t)$. The event $B'_\epsilon(t)$ has exponentially small probability, as it belongs to the event $C'_\epsilon(t)$, the probability of which can be estimated via Poisson distribution:

$$P(C'_\epsilon(t)) \leq C'_0 e^{-h'(\epsilon)t} \quad (11)$$

where the constant C'_0 depends on λ and ϵ , $h'(\epsilon) > 0$.

The event $D'_\epsilon(t)$ belongs to $J'_\epsilon(t)$, that at least one particle of those initially in the interval $(0, c'_\epsilon t\delta + (v - \epsilon)t)$ was not absorbed by the boundary during time t . The probability of event $J'_\epsilon(t)$ can be estimated from above by the expectation of the number of particles, that were initially inside the interval $(0, c'_\epsilon t\delta + (v - \epsilon)t)$, and deviated more than on ϵt its mean assignment point (if at time zero the particle is at some point x , then its assignment point is $x - vt$) at time t :

$$P(J'_\epsilon(t)) \leq \lambda \int_0^{c'_\epsilon t\delta + (v - \epsilon)t} p'(t, x) dx$$

where $p'(t, x)$ is the probability that a particle started from x , $x < c'_\epsilon t\delta + (v - \epsilon)t$, at time t will be outside the interval $(x - vt - \epsilon t, x - vt + \epsilon t)$, that is

$$p'(t, x) = 1 - \frac{1}{\sqrt{2\pi t}} \int_{x - (v + \epsilon)t}^{x - (v - \epsilon)t} e^{-\frac{(y - x + vt)^2}{2t}} dy$$

After substituting $z = \frac{y - x + vt}{\sqrt{t}}$, we find

$$p'(t, x) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\epsilon\sqrt{t}}^{\epsilon\sqrt{t}} e^{-\frac{z^2}{2}} dz = 2\bar{\Phi}(\epsilon\sqrt{t})$$

Thus

$$P(J'_\epsilon(t)) \leq C_\epsilon t e^{-\frac{\epsilon^2 t}{2}}$$

Taking into account (11), we get that for any sufficiently small $\epsilon > 0$

$$P\left(N(t) < \frac{(\lambda - \epsilon)(v - \epsilon)t}{1 - (\lambda - \epsilon)\delta}\right) = O\left(e^{-\alpha_2(\epsilon)t}\right) \quad (12)$$

for some $\alpha_2(\epsilon) > 0$, together with (10) this proves the first assertion of the theorem.

3.2 Zero drift

Here we prove the second assertion of the Theorem.

We start with “trivial” estimate from below, we mean by this that we estimate just the number of particle collisions with the fixed boundary. Consider an auxiliary model where the boundary does not change its position while hit by particles but the particles disappear at the first collision with the boundary. Let $L(t)$ be the number of collisions with the boundary for this model during time t . Evidently, we have $L(t) \leq N(t)$ a. s. Denote $K_0(t) = K_0(t, d)$ the number of particles, which did not reach 0 during time t and at time t were to the left of the point $d\sqrt{t}$, where $d > 0$. Put $K(t) = N(t) - L(t)$. If the condition $\delta L(t) \geq d\sqrt{t}$ holds then $K(t) \geq K_0(t)$, as in the basic model all particles, situated at time t inside $[0, \delta L(t)]$, would have been absorbed by the moving boundary. Thus one can state that

$$MK(t) \geq MK_0(t)I(L(t) \geq d\sqrt{t}) \quad (13)$$

where $I(L(t) \geq d\sqrt{t})$ is the indicator of the corresponding event.

Let us find the joint distribution of the variables $L(t)$ and $K_0(t)$. We begin with the means. Let $p(t, x)$ be the probability that a particle situated initially at x , during time t will reach the boundary:

$$p(t, x) = P(\min_{s \leq t} (x + w(s)) \leq 0)$$

As

$$\begin{aligned} P(\min_{s \leq t} (x + w(s)) \leq 0) &= P(\min_{s \leq t} w(s) \leq -x) \\ &= P(\max_{s \leq t} w(s) \geq x) \end{aligned}$$

then by formula (7) with $v = 0$ we have

$$p(t, x) = 1 - \Phi\left(\frac{x}{\sqrt{t}}\right) + \Phi\left(-\frac{x}{\sqrt{t}}\right) = 2\bar{\Phi}\left(\frac{x}{\sqrt{t}}\right)$$

Let us show that for zero drift ($v = 0$)

$$ML(t) = \lambda \sqrt{\frac{2t}{\pi}}$$

In fact, we have

$$\begin{aligned} ML(t) &= \lambda \int_0^\infty p(t, x) dx = 2\lambda \int_0^\infty \bar{\Phi}\left(\frac{x}{\sqrt{t}}\right) dx \\ &= 2\lambda \sqrt{t} \int_0^\infty \bar{\Phi}(x) dx = \lambda \sqrt{\frac{2t}{\pi}} \end{aligned}$$

where we integrated the latter integral by parts

$$\int_0^\infty \bar{\Phi}(x) dx = \int_0^\infty x d\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty x e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}}$$

Let us find $MK_0(t)$. Denote $\sigma(t, x)$ the probability that a particle, situated initially at x , during time t will not reach 0 and moreover at time t it will be to the left of $d\sqrt{t}$. Otherwise speaking

$$\sigma(t, x) = P(x + \min_{[0, t]} w_s > 0, x + w_t < d\sqrt{t}) = P(\max_{[0, t]} w_s < x, w_t > x - d\sqrt{t})$$

The probability $\sigma(t, x)$ can be written as well in the following way

$$\sigma(t, x) = P(w_t > x - d\sqrt{t}) - 2P(w_t > x) + P(\max_{[0, t]} w_s > x, w_t < x - d\sqrt{t})$$

where due to [2], (see p. 45), we have

$$\begin{aligned}
P(\max_{[0,t]} w_s > x, w_t < x - d\sqrt{t}) &= \sqrt{\frac{2}{\pi t^3}} \int_{-\infty}^{x-d\sqrt{t}} da \int_x^{\infty} (2b-a) e^{-(2b-a)^2/2t} db \\
&= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{x-d\sqrt{t}} e^{-(2x-u)^2/2t} du \\
&= \bar{\Phi}\left(\frac{x+d\sqrt{t}}{\sqrt{t}}\right)
\end{aligned}$$

Using this formula we find

$$\sigma(t, x) = \bar{\Phi}\left(\frac{x-d\sqrt{t}}{\sqrt{t}}\right) - 2\bar{\Phi}\left(\frac{x}{\sqrt{t}}\right) + \bar{\Phi}\left(\frac{x+d\sqrt{t}}{\sqrt{t}}\right)$$

and then

$$MK_0(t) = \int_0^{\infty} \sigma(t, x) dx = \sqrt{t} \left(\int_{-d}^{\infty} \bar{\Phi}(x) dx - 2 \int_0^{\infty} \bar{\Phi}(x) dx + \int_d^{\infty} \bar{\Phi}(x) dx \right) = \sigma\sqrt{t}$$

where the constant

$$\sigma = \int_{-d}^{\infty} \bar{\Phi}(x) dx - 2 \int_0^{\infty} \bar{\Phi}(x) dx + \int_d^{\infty} \bar{\Phi}(x) dx = \int_{-d}^0 \bar{\Phi}(x) dx - \int_0^d \bar{\Phi}(x) dx$$

Lemma 4 *Random variables $L(t)$ and $K_0(t)$ are independent. $L(t)$ has Poisson distribution with parameter $\lambda\sqrt{\frac{2t}{\pi}}$, and $K_0(t)$ has Poisson distribution with parameter $\sigma\sqrt{t}$.*

Proof. Let $\pi_n(a)$ be Poisson-distribution with parameter λa , $L(a, t)$ be the number of particles, which were inside $(0, a)$ at time $t = 0$ and during t were absorbed by the boundary, $K_0(a, t)$ be the number of particles which were inside $(0, a)$ at time $t = 0$, during time t did not reach 0 and at time t were to the left of the point $d\sqrt{t}$. Put

$$q(t, x) = 1 - p(t, x) - \sigma(t, x)$$

Find the joint distribution of the random variables $L(a, t)$ and $K_0(a, t)$:

$$P(L(a, t) = k, K_0(a, t) = m) = \sum_{n=k+m}^{\infty} \pi_n(a) a^{-n} \sum \int_0^a \dots \int_0^a \prod_{l=1}^k p(t, x_{i_l}) dx_{i_l} \prod_{l=1}^m \sigma(t, x_{j_l}) dx_{j_l} \prod_{l=1}^{n-k-m} q(t, x_{r_l}) dx_{r_l}$$

where the inner sum is over all ordered arrays $i_1 < \dots < i_k$ and $j_1 < \dots < j_m$, of length k and m correspondingly, such that $\{i_1 < \dots < i_k\} \cap \{j_1 < \dots < j_m\} = \emptyset$. Here

$$\{r_1 < \dots < r_{n-k-m}\} = \{1, \dots, n\} \setminus \{i_1 < \dots < i_k\} \cup \{j_1 < \dots < j_m\}$$

As the inner sum consists of

$$\frac{n!}{k!m!(n-k-m)!}$$

equal terms $(\hat{p}(t, a))^k (\hat{\sigma}(t, a))^m (\hat{q}(t, a))^{n-k-m}$, where $\hat{p}(t, a) = \int_0^a p(t, x) dx$, $\hat{\sigma}(t, a) = \int_0^a \sigma(t, x) dx$, $\hat{q}(t, a) = \int_0^a q(t, x) dx$, then

$$\begin{aligned}
P(L(a, t) = k, K_0(a, t) = m) &= \sum_{n=k+m}^{\infty} \pi_n(a) a^{-n} \frac{n!}{k!m!(n-k-m)!} (\hat{p}(t, a))^k (\hat{\sigma}(t, a))^m (\hat{q}(t, a))^{n-k-m} \\
&= \frac{(\lambda \hat{p}(t, a))^k (\lambda \hat{\sigma}(t, a))^m}{k! m!} e^{-\lambda a} \sum_{n=k+m}^{\infty} \frac{(\lambda \hat{q}(t, a))^{n-k-m}}{(n-k-m)!} \\
&= \frac{(\lambda \hat{p}(t, a))^k e^{-\lambda \hat{p}(t, a)}}{k!} \frac{(\lambda \hat{\sigma}(t, a))^m e^{-\lambda \hat{\sigma}(t, a)}}{m!}
\end{aligned}$$

To get joint distribution of $L(t)$ and $K_0(t)$, let us tend a to infinity. In the limit we will get direct product of Poisson distributions with parameters $\lambda\sqrt{\frac{2t}{\pi}}$ and $\sigma\sqrt{t}$, as $\hat{p}(t, a) \rightarrow \sqrt{\frac{2t}{\pi}}$ and $\hat{\sigma}(t, a) \rightarrow \sigma\sqrt{t}$ for $a \rightarrow \infty$. The lemma is proved.

Let us prove now that the constant in the lower bound is in fact greater than the trivial one. Choose $d > 0$ so that $d < \delta ML(t)/\sqrt{t}$. By (13) and independence of random variables $L(t)$ and $K_0(t)$, we have

$$MK(t) \geq MK_0(t)P(L(t) \geq d\sqrt{t})$$

and

$$MK_0(t)P(L(t) \geq d\sqrt{t}) = \sigma\sqrt{t} - MK_0(t)P(L(t) < d\sqrt{t})$$

As, for a given choice of d , the probability of the event $L(t) < d\sqrt{t}$ is exponentially small

$$P(L(t) < d\sqrt{t}) \leq C_4 e^{-\phi\sqrt{t}}$$

then one can assert that $MK_0(t)P(L(t) \geq d\sqrt{t}) \geq \sigma_0\sqrt{t}$ for some constant $\sigma_0 > 0$, where $\sigma_0 < \sigma$.

Prove now the bound from above in the second statement of the Theorem. Our goal is to estimate the probability $P(N(t) > x\sqrt{t})$ from above. Take arbitrary $\epsilon > 0$, so that $(\lambda + \epsilon)\delta < 1$. Subdivide the event $A(t, N) = \{N(t, \omega) = N\}$ into the union of events

$$A(t, N) = \cup_{n \leq N} B_0(N, n, t)$$

where $B_0(N, n, t)$ is the event that the boundary absorbed exactly n particles of the particles, situated initially inside $(0, N\delta)$, and exactly $m = N - n$ particles of the particles, situated initially inside $(N\delta, \infty)$. Denote $B_{0,\epsilon}(N, t)$ the union of events $B_0(N, n, t)$ with $n > (\lambda + \epsilon)\delta N$. The event $B_{0,\epsilon}(N, t)$ has exponentially small probability, as this event belongs to the event $C_{0,\epsilon}(N)$, that at time 0 inside $(0, N\delta)$ there was no more than $(\lambda + \epsilon)\delta N$ particles. The probability of the latter event can be estimated by the Poisson distribution:

$$P(C_{0,\epsilon}(N)) \leq C_0 e^{-h(\epsilon)N} \quad (14)$$

where $h(\epsilon) > 0$ and C_0 is some constant depending only on λ, ϵ .

Denote $D_{0,\epsilon}(N, t)$ the union of events $B_0(N, n, t)$ with $n < (\lambda + \epsilon)\delta N$. From the considerations above it follows that the event $D_{0,\epsilon}(N, t)$ is equivalent to the event

$$m = N - n > N(1 - (\lambda + \epsilon)\delta)$$

At the same time the event $m > N(1 - (\lambda + \epsilon)\delta)$ belongs to the event that among the particles which were at time 0 to the right of the point $N\delta$, more than $N(1 - (\lambda + \epsilon)\delta)$ particles during time t were at least once to the left of the point $N\delta$. Denote the latter event by $J_{0,\epsilon}(N, t)$, and denote $\kappa(t)$ the number of particles which were at time 0 to the right of the point $N\delta$ but having visited the left side of $N\delta$ at least once during time t (that is $P(J_{0,\epsilon}(N, t)) = P(\kappa(t) > N(1 - (\lambda + \epsilon)\delta))$). The distribution of this random variable does not depend on N and coincides with the distribution of the random variable $L(t)$, defined above. By lemma 4 and the properties of Poisson distribution, we have the following estimate

$$P(\kappa(t) > y\sqrt{t}) \leq C_1 e^{-\beta\sqrt{t}} \quad (15)$$

valid for some $\beta > 0$, $y \geq y_0 > \lambda\sqrt{\frac{2}{\pi}}$ and constant C_1 , depending on y_0 .

As the event $A(t, N)$ belongs to the union of the events $C_{0,\epsilon}(N)$ and $J_{0,\epsilon}(N, t)$, then

$$P(N(t) > x\sqrt{t}) \leq \sum_{N > x\sqrt{t}} P(C_{0,\epsilon}(N)) + \sum_{N > x\sqrt{t}} P(J_{0,\epsilon}(N, t))$$

By (14), for the first sum the following estimate holds

$$\sum_{N > x\sqrt{t}} P(C_{0,\epsilon}(N)) = O\left(e^{-h(\epsilon)x\sqrt{t}}\right)$$

The second sum, by (15), has the following estimate

$$\sum_{N > x\sqrt{t}} P(J_{0,\epsilon}(N, t)) = \sum_{N > x\sqrt{t}} P(\kappa(t) > (1 - (\lambda + \epsilon)\delta)N) \leq C_2 e^{-\beta(\epsilon)\sqrt{t}}$$

for some constant $\beta(\epsilon) > 0$, $x > x_0 = y_0(1 - (\lambda + \epsilon)\delta)^{-1}$ and some constant C_2 , depending on x_0 . It follows, that for sufficiently large x

$$P(N(t) > x\sqrt{t}) \leq C_3 e^{-\gamma x\sqrt{t}}$$

Hence, for t sufficiently large

$$MN(t) \leq C\sqrt{t}$$

The Theorem is proved.

Proof of the second part of Lemma 1 Consider first the case of zero drift. Let us find the distribution of particle configuration at the time of first hitting the boundary. Put

$$\sigma(t, x, y)dy = P(x + w_t \in dy, x + \min_{s \leq t} w_t > 0), x > 0, y > 0$$

Thus, $\sigma(t, x, y)dy$ is the probability that the particle which were at time $t = 0$ at $x > 0$, during time t will not reach 0 and will be in the interval $(y, y + dy)$ at time t . As the distribution of w_t coincides with the distribution of $-w_t$, we have

$$\sigma(t, x, y)dy = P(x - w_t \in dy, x - \max_{s \leq t} w_t > 0) = P(w_t \in d(x - y), \max_{s \leq t} w_t < x)$$

To calculate $\sigma(t, x, y)$ we will use the following formula for the joint distribution (see, for example [2])

$$P(w_t \in da, \max_{s \leq t} w_s \in db) = \sqrt{\frac{2}{\pi t^3}} (2b - a) e^{-(2b-a)^2/2t} da db, 0 \leq b, b \geq a$$

The change of variables $u = (2b - x + y)^2/2t$ gives

$$\begin{aligned} \sigma(t, x, y) &= \sqrt{\frac{2}{\pi t^3}} \int_{\max(x-y, 0)}^x (2b - x + y) e^{-(2b-x+y)^2/2t} db \\ &= \sqrt{\frac{1}{2\pi t}} \int_{(x-y)^2/2t}^{(x+y)^2/2t} e^{-u} du \\ &= \sqrt{\frac{1}{2\pi t}} \left(e^{-(x-y)^2/2t} - e^{-(x+y)^2/2t} \right) \end{aligned}$$

Let τ be the moment of first hitting the boundary by one of the particles.

Lemma 5 *The probability density of the random variable τ is*

$$P(\tau \in dt) = \frac{\lambda}{\sqrt{2\pi t}} e^{-\lambda\sqrt{2t/\pi}} dt$$

Under the condition $\tau = t$ the particle configuration at time t is an inhomogeneous Poisson point field with the rate

$$\lambda\psi(t, y) = \lambda \int_0^\infty \sigma(t, x, y) dx = \lambda(\Phi(y/\sqrt{t}) - \Phi(-y/\sqrt{t}))$$

where $\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-z^2/2} dz$.

Proof. Let B_1, \dots, B_l be an array consisting of l pairwise non intersecting intervals on the positive axis. Let us find the joint distribution $P(\tau \in dt, \eta(t, B_i) = k_i, i = 1, \dots, l)$, where $\eta(t, B_i)$ is the number of particles in the interval B_i at time t . Let $\pi_n(a)$ be the Poisson distribution with parameter λa , $\eta(a, t, B_i)$ be the number

of particles, which were initially inside the interval $(0, a)$ and at time t were in B_i . Let $\tau(a)$ be the first time moment, when one of the particles, which were in $(0, a)$ at time $t = 0$, hits the boundary. Denote $g(t, x)$ the probability density of the time β_x of first hitting the point 0 by the particle under the condition that at time $t = 0$ this particle were at x . It is well known that (see for example [2]), that

$$g(t, x) = \frac{x}{\sqrt{2\pi t^3}} e^{-x^2/2t}$$

Let us find the joint distribution of random variables $\eta(a, t, B_i), i = 1, \dots, l$ and $\tau(a)$:

$$\begin{aligned} P(\tau(a) \in dt, \eta(a, t, B_i) = k_i, i = 1, \dots, l) &= \sum_{n=1+k_1+\dots+k_l}^{\infty} \pi_n(a) a^{-n} \sum \int_0^a \dots \int_0^a g(t, x_m) dt dx_m \\ &\quad \prod_{j=1}^l \prod_{s=1}^{k_j} dx_{i_{j,s}} \int_{y \in B_j} \sigma(t, x_{i_{j,s}}, y) dy \\ &\quad \prod_{s=1}^{n-1-k_1-\dots-k_l} dx_{r_s} \int_{y \in R_+ \setminus \cup_{j=1}^l B_j} \sigma(t, x_{r_s}, y) dy \end{aligned}$$

where the inner sum is over all pairwise non intersecting ordered arrays $m, i_{1,1} < \dots < i_{1,k_1}, i_{2,1} < \dots < i_{2,k_2}, \dots, i_{l,1} < \dots < i_{l,k_l}$ of the lengths $1, k_1, k_2, \dots, k_l$ correspondingly. Here

$$\{r_1 < \dots < r_{n-1-k_1-\dots-k_l}\} = \{1, \dots, n\} \setminus (\{m\} \cup \{i_{1,1} < \dots < i_{1,k_1}\} \cup \dots \cup \{i_{l,1} < \dots < i_{l,k_l}\})$$

As the inner sum consists of

$$\frac{n!}{k_1! \dots k_l! (n-1-k_1-\dots-k_l)!}$$

equal terms $\hat{g}(a, t) dt \prod_{j=1}^l (\hat{\sigma}(a, t, B_j))^{k_j} (\hat{\sigma}(a, t, R_+ \setminus \cup_{j=1}^l B_j))^{n-1-k_1-\dots-k_l}$, where

$$\begin{aligned} \hat{g}(t, a) &= \int_0^a g(t, x) dx \\ \hat{\sigma}(a, t, B_j) &= \int_0^a dx \int_{y \in B_j} \sigma(t, x, y) dy \end{aligned}$$

then

$$\begin{aligned} P(\tau(a) \in dt, \eta(a, t, B_i) = k_i, i = 1, \dots, l) &= \sum_{n=1+k_1+\dots+k_l}^{\infty} \pi_n(a) a^{-n} \frac{n!}{k_1! \dots k_l! (n-1-k_1-\dots-k_l)!} \\ &\quad \hat{g}(a, t) dt \prod_{j=1}^l (\hat{\sigma}(a, t, B_j))^{k_j} (\hat{\sigma}(a, t, R_+ \setminus \cup_{j=1}^l B_j))^{n-1-k_1-\dots-k_l} \\ &= \lambda \hat{g}(a, t) dt \prod_{j=1}^l \frac{(\lambda \hat{\sigma}(a, t, B_j))^{k_j}}{k_j!} e^{-\lambda a} \times \\ &\quad \times \sum_{n=1+k_1+\dots+k_l}^{\infty} \frac{(\lambda \hat{\sigma}(a, t, R_+ \setminus \cup_{j=1}^l B_j))^{n-1-k_1-\dots-k_l}}{(n-1-k_1-\dots-k_l)!} \\ &= \lambda \hat{g}(a, t) dt \prod_{j=1}^l \frac{(\lambda \hat{\sigma}(a, t, B_j))^{k_j}}{k_j!} e^{-\lambda a + \lambda \hat{\sigma}(a, t, R_+ \setminus \cup_{j=1}^l B_j)} \end{aligned}$$

Taking into account that

$$\sum_{j=1}^l \hat{\sigma}(a, t, B_j) + \hat{\sigma}(a, t, R_+ \setminus \cup_{j=1}^l B_j) = \int_0^a P(\beta_x > t) dx = a - \int_0^a P(\beta_x < t) dx$$

we have

$$-a + \hat{\sigma}(a, t, R_+ \setminus \cup_{j=1}^l B_j) = - \sum_{j=1}^l \hat{\sigma}(a, t, B_j) - \int_0^a P(\beta_x < t) dx$$

Thus

$$P(\tau(a) \in dt, \eta(a, t, B_i) = k_i, i = 1, \dots, l) = \lambda \hat{g}(a, t) e^{-\int_0^a P(\beta_x < t) dx} dt \prod_{j=1}^l \frac{(\lambda \hat{\sigma}(a, t, B_j))^{k_j}}{k_j!} e^{-\lambda \hat{\sigma}(a, t, B_j)}$$

To obtain the joint distribution of τ and $\eta(t, B_i)$, we tend a to infinity. Then

$$\begin{aligned} \hat{g}(a, t) &\rightarrow \int_0^\infty g(t, x) dx = \sqrt{\frac{1}{2\pi t}} \\ \int_0^a P(\beta_x < t) dx &\rightarrow \int_0^\infty P(\beta_x < t) dx = \sqrt{\frac{2t}{\pi}} \\ \hat{\sigma}(a, t, B_j) &\rightarrow \hat{\sigma}(t, B_j) = \int_{y \in B_j} dy \int_0^\infty \sigma(t, x, y) dx \\ &= \int_{y \in B_j} dy \psi(t, y) \end{aligned}$$

where $\psi(t, y) = \Phi(y/\sqrt{t}) - \Phi(-y/\sqrt{t})$. Hence

$$P(\tau \in dt, \eta(t, B_i) = k_i, i = 1, \dots, l) = \frac{\lambda}{\sqrt{2\pi t}} e^{-\lambda \sqrt{\frac{2t}{\pi}}} dt \prod_{j=1}^l \frac{(\lambda \hat{\sigma}(t, B_j))^{k_j}}{k_j!} e^{-\lambda \hat{\sigma}(t, B_j)}$$

The lemma is proved.

Let us prove that the random variable k_1 can take infinite values with positive probability. Assume that $\tau = t$. Denote m_1 the number of particles in the interval $[0, \delta]$ at time $\tau = t$, m_2 the number of particles at time $\tau = t$ in the interval $D_1 = [\delta, (m_1 + 1)\delta]$, if $m_1 > 0$, and denote m_3 the number of particles at time $\tau = t$ in the interval $D_2 = [(m_1 + 1)\delta, (m_1 + 1)\delta + m_2\delta]$, if $m_1 > 0$, $m_2 > 0$, etc. The sequence of pairs $\mu_i = (m_{i-1}, m_i)$ is a discrete time homogeneous Markov chain. Transition probabilities are

$$p_{\mu_i \mu_{i+1}} = \frac{(\lambda \hat{\sigma}(t, D_i))^{m_{i+1}}}{m_{i+1}!} e^{-\lambda \hat{\sigma}(t, D_i)}$$

The states $(m, 0)$ are absorbing. Random variable k_1 is defined by the random stopping time of the Markov chain μ_i . The boundary stops, when the chain hits some of the absorbing states. We introduce Lyapounov function $f(\mu_i) = m_i$. Its one step increment equals

$$\begin{aligned} M(f(\mu_{i+1}) - f(\mu_i) | \mu_i = (m_{i-1}, m_i)) &= \lambda \hat{\sigma}(t, D_i) - m_i \\ &= \lambda \int_{(m_{i-1}+1)\delta}^{(m_{i-1}+1)\delta + m_i\delta} \psi(t, y) dy - m_i \end{aligned}$$

As $\psi(t, y) \rightarrow 1$ for $y \rightarrow \infty$ and $\lambda\delta > 1$, then there exists such $\epsilon > 0$, that for sufficiently large m_{i-1}

$$M(f(\mu_{i+1}) - f(\mu_i) | \mu_i = (m_{i-1}, m_i)) \geq \epsilon m_i$$

This is the transience of the Markov chain μ_i . Thus, $P(k_1 = \infty) > 0$.

The case of nonzero drift is considered similarly. Moreover, as we consider the lower bound, it is intuitively clear that the drift can only increase the probability that the boundary explodes to infinity.

References

- [1] A. N. Shiryaev. Foundations of stochastic financial mathematics. V. 2, Moscow 1998.
- [2] K. Ito, G. McKean. Diffusion processes and their trajectories. 1968.
- [3] R. L. Dobrushin. About Poisson law of particle distribution in space. Ukrainian Math. Journal, 1956, v. 8, No. 2, 130-134.